Analytic Pricing of Convertible Bonds

Colin Turfus

Initial Version: December 6, 2016
Current Version: December 31, 2017

Abstract

We consider the problem of pricing convertible bonds deriving an analytic solution for the value of an option to convert the bond to equity at the bond's face value at bond maturity. The formula derived is based on the perturbation expansion techniques developed by Turfus (2016a) and Turfus (2016b). The main result is an analytic expression for the correction to the value of the option arising from the expected negative correlation between the equity price and the credit spread for the bond issuer. Somewhat counter-intuitively the correlation risk is seen to be wrong-way in the sense that it *decreases* the value of the embedded optionality. Although the method assumes that the impact of the credit intensity is weak, it is proposed that it ought to give reasonable approximations even for realistic market values.

1 Introduction

We look to price the embedded European option associated with a convertible bond which allows the holder to exchange the bond for a fixed number of the issuer's shares at maturity, in lieu of repayment of notional. Our model is closely related to one proposed by Davis and Lischka (1999), specifically the variant in their §2.3 where interest rates are taken to be deterministic but hazard rates stochastic. They consider the more general problem of Bermudan options to convert during the life of the bond and adopt a tree-based approach to facilitate the calculation. They use a Hull and White (1990) short rate model to enable greater tractability in their numerical computations, whereas we prefer to adopt a Black and Karasinski (1991) model, ensuring as it does that hazard rates remain positive. While they take into account correlation between the equity and the default process, they do not take into account the default (jump) risk on the equity process which results in an increased drift, as we shall see, when the option to convert is priced contingent on no default having occurred. Later authors such as Ayache et al. (2003) who look to handle the default contingence of options to convert more carefully similarly neglect the modification which is needed to the equity dynamics when it is modelled contingent on no default.

We set out in section 2 below a description of the underlying stochastic model for the equity and default intensity processes, going on to show how a PDE describing the value of an option to convert a bond to equity at face value can be derived. In section 3 we show how, under an asymptotic assumption of a weak default intensity, an analytic expression for this can be inferred as a perturbation expansion. Our main result is expressed in the concluding section 4, in particular in Theorem 4.1.

2 Model Description

We consider a credit default process driven by a stochastic default intensity, the dynamics of which are governed by a Black-Karasinski short rate model. The equity price we take to be given by a jump-diffusion process, with a downward jump to zero at the time of default. The diffusive processes are mutually correlated. We follow Turfus and Schubert (2017) in their related calculation for contingent convertible (CoCo) prices

in assuming from the outset that rates are deterministic. Thus we propose:

$$\frac{d\lambda_t}{\lambda_t} = -\alpha_\lambda(t)(\ln\lambda_t - \ln\theta_\lambda(t))\,dt + \sigma_\lambda(t)dW_t^1, \tag{1}$$

$$\frac{dS_t}{S_t} = (\overline{r}(t) - q(t) + \lambda_t)\,dt + \sigma_S(t)\,dW_t^2 - dn_t. \tag{2}$$

for $t < T_m = \min(\tau, T)$, where τ is the time of default of the bond/equity issuer and T is the maturity of the convertible bond under consideration. Here λ_t is the default intensity with mean reversion rate $\alpha_\lambda(t)$ and mean reversion level $\theta_\lambda(t)$ chosen to satisfy the no-arbitrage condition that

$$E\left[e^{-\int_0^t \lambda_s\,ds}\right] = e^{-\int_0^t \overline{\lambda}(s)\,ds} \tag{3}$$

under the martingale (money market) measure for $0 < t \leq T$, with $\overline{\lambda}(t)$ the instantaneous forward default intensity observed at the initial time $t = 0$. Here S_t is the equity price, $\overline{r}(t)$ is the instantaneous forward rate of interest, $q(t)$ is the expected (continuous) dividend rate, dW_t^1 and dW_t^2 are correlated Brownian motions with

$$\mathrm{corr}(W_t^1, W_t^2) = \rho_{\lambda S}$$

and n_t is a Cox process with intensity λ_t giving rise to an equity price jump to zero, contingent on default.

We suppose that the convertible bond is a standard bond priceable by the usual means, with an embedded option to exchange the notional repayment at T for a fixed number M of ordinary shares per unit notional. We write the PV of a convertible bond with notional N as

$$V(t) = Nf_{\mathrm{bond}}(t) + Nf_{\mathrm{option}}(t). \tag{4}$$

where $f_{\mathrm{bond}}(t)$ expresses the part of the PV generated by bond cash flows, i.e. the coupon payments and notional repayment, and $f_{\mathrm{option}}(t)$ expresses the additional contribution to the PV resulting from the embedded option. We focus here on the calculation of the latter term. Making explicit the dependency on the market variable S_t and λ_t, we denote $f_{\mathrm{option}}(t) \equiv f(S_t, \lambda_t, t)$. We deduce by standard means that $f(S, \lambda, t)$ will be governed by the following backward diffusion equation:

$$\begin{aligned}\frac{\partial f}{\partial t} + (\overline{r}(t) - q(t) + \lambda)S\frac{\partial f}{\partial S} - \alpha_\lambda(t)(\ln\lambda - \ln\theta_\lambda(t))\lambda\frac{\partial f}{\partial\lambda} + \frac{1}{2}\left(\sigma_S^2 S^2\frac{\partial^2 f}{\partial S^2} + 2\rho_{\lambda S}\sigma_S\sigma_\lambda S\lambda\frac{\partial^2 f}{\partial S\partial\lambda} + \sigma_\lambda^2\lambda^2\frac{\partial^2 f}{\partial\lambda^2}\right)\\ = (\overline{r}(t) + \lambda)f\end{aligned} \tag{5}$$

for $t \leq T_m$ and subject to the assumed terminal condition that

$$f(S_T, \lambda_T, T) = \max\{MS_T - 1, 0\}$$

with M the number of shares issued per unit notional. We seek to calculate $f(S_0, \lambda_0, 0)$, the value at time $t = 0$ of the embedded option to convert.

3 Perturbation Analysis

In the absence of an exact analytic solution to (5), we seek an approximate solution. The perturbation approach used here to obtain convertible bond prices was first proposed by Turfus (2016a), who proposed that the short rate in a Black and Karasinski (1991) model be considered weak in an asymptotic sense. It is closely related to that used by Turfus (2016b) in calculating the prices of CoCo bonds under an assumed Black-Karasinski short rate model for the conversion intensity, combined with an equity jump-diffusion process. This was an extension of the earlier calculation of Turfus and Schubert (2017) based on an assumed

Hull and White (1990) short rate model. However in both these CoCo bond calculations an assumption was made that the credit intensity *volatility* was small, rather than the credit intensity itself, as proposed below. Particularly in the Black-Karasinski case the former approximation leads to less accurate results, whence we suggest the approach set out below is superior.

Following Turfus (2016a), we suppose that the default intensity is weak, i.e. that an asymptotic parameter

$$\epsilon = \frac{\int_0^T \overline{\lambda}(t)\,dt}{\int_0^T \alpha_\lambda(t)\,dt}$$

is small, wherefore we define a scaled forward intensity $\tilde{\lambda}(t) = \frac{1}{\epsilon}\overline{\lambda}(t)$. We proceed by noting that the solution to (1) can be written

$$\ln \lambda_t = \ln \lambda^*(t) + \int_0^t \phi(s,t)\sigma_\lambda(s)dW_s^1$$

where

$$\phi(s,t) := e^{-\int_s^t \alpha_\lambda(u)du} \tag{6}$$

for some as yet unknown $O(\epsilon)$ function $\lambda^*(t)$. For computational convenience we propose a new stochastic intensity coordinate

$$y_t := \ln \lambda_t - \ln \lambda^*(t) + \tfrac{1}{2} I_\lambda(0,t) \tag{7}$$

with

$$I_\lambda(t_1,t_2) := \int_{t_1}^{t_2} \phi^2(v,t_2)\,\sigma_\lambda^2(v)\,dv, \tag{8}$$

We also write

$$\lambda^*(t) = \epsilon\lambda_1^*(t) + \epsilon^2\lambda_2^*(t) + O(\epsilon^3)$$

where we know from Turfus (2016a) that the no-arbitrage condition (3) and the initial condition $y_0 = 0$ require us to choose[1]

$$\lambda_1^*(t) = \tilde{\lambda}(t) \tag{9}$$

$$\lambda_2^*(t) = \tilde{\lambda}(t)\int_0^t \left(F_\lambda(v,t) - 1\right)\tilde{\lambda}(v)\,dv. \tag{10}$$

with

$$F_\lambda(v,t) := \exp\left(\phi(v,t) I_\lambda(0,v)\right). \tag{11}$$

Similarly, we define a new characteristic stochastic equity process x_t such that

$$S_t = F(t)\mathcal{E}(x_t), \tag{12}$$

where $\mathcal{E}(X_t) := \exp\left(X_t - \frac{1}{2}[X]_t\right)$ is a Doléans-Dade exponential with $[X]_t$ the quadratic variation, and

$$F(t) := S_0 e^{\int_0^t (\overline{r}(s) - q(s) + \overline{\lambda}(s))ds}. \tag{13}$$

We define also for notational convenience an equity payoff function

$$M_0(x,t) = MF(t)e^{x - \frac{1}{2}I_S(0,t)},$$

where[2]

$$I_S(t_1,t_2) = \int_{t_1}^{t_2} \sigma_S^2(u)du. \tag{14}$$

[1] We will not in fact use the expression for $\lambda_2^*(t)$ to the level of approximation explored in the present context.

[2] Note that $I_S(0,t)$ is the quadratic variation of x_t required for (12).

Changing to the new coordinates (x, y, t) and expressing

$$h(x, y, t) \equiv f(S, \lambda, t),$$

(5) can be rewritten as:

$$\mathcal{L}[h] \sim (\lambda^*(t) e^{y - \frac{1}{2} I_\lambda(0,t)} - \epsilon\tilde{\lambda}(t)) \left(h - \tfrac{\partial h}{\partial x}\right), \tag{15}$$

with final condition

$$h(x_T, y_T, T) = \max\{M_0(x_T, T) - 1, 0\},$$

here $\mathcal{L}[.]$ is a standard forced diffusion operator given by

$$\mathcal{L}[.] = \frac{\partial}{\partial t} - \alpha_\lambda(t) y \frac{\partial}{\partial y} + \frac{1}{2}\left(\sigma_S^2(t)\frac{\partial^2}{\partial x^2} + 2\rho_{\lambda S}\sigma_S(t)\sigma_\lambda(t)\frac{\partial^2 h}{\partial x \partial y} + \sigma_\lambda^2(t)\frac{\partial^2}{\partial y^2}\right) - (\overline{r}(t) + \overline{\lambda}(t)). \tag{16}$$

To solve this we pose an asymptotic expansion

$$h(x, y, t) = h_0(x, y, t) + \epsilon h_1(x, y, t) + O(\epsilon^2). \tag{17}$$

At zeroth order we must solve:

$$\mathcal{L}[h_0] = 0$$

with final condition

$$h_0(x_T, y_T, T) = \max\{M_0(x_T, T) - 1, 0\}.$$

This can be achieved by means of a Green's function

$$G(x, y, t; \xi, \eta, v) = B(t, v) H(v - t) \frac{\partial^2}{\partial \xi \partial \eta} N_2(\xi - x, \eta - y\phi(t, v); R(t, v)). \tag{18}$$

where $H(.)$ is the Heaviside step function,

$$B(t, v) = e^{-\int_t^v (\overline{r}(s) + \overline{\lambda}(s)) ds}, \tag{19}$$

is a risky discount factor, and $N_2(x, y, R(t, v))$ is a bivariate Gaussian probability distribution function with mean $\mathbf{0}$ and covariance matrix

$$R(t, v) = \begin{pmatrix} I_S(t, v) & I_R(t, v) \\ I_R(t, v) & I_\lambda(t, v) \end{pmatrix} \tag{20}$$

where

$$I_R(t_1, t_2) = \rho_{\lambda S} \int_{t_1}^{t_2} \phi(v, t_2)\sigma_\lambda(v)\sigma_S(v)\, dv. \tag{21}$$

Making use of (18) we obtain

$$\begin{aligned} h_0(x_t, y_t, t) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G(x_t, y_t, t; \xi, \eta, T) \max\{M_0(\xi, T), 1\}\, d\xi\, d\eta \\ &= B(t, T)\left(\mathcal{E}(x_t) MF(T) N(d_1(x_t, t, T)) - N(d_2(x_t, t, T))\right) \end{aligned} \tag{22}$$

where

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp\left(-\tfrac{1}{2}u^2\right) du, \tag{23}$$

$$d_2(x, t, v) = \frac{\ln M_0(x, v) - \ln K}{\sqrt{I_S(t, v)}}, \tag{24}$$

$$d_1(x, t, v) = d_2(x, t, v) + \sqrt{I_S(t, v)}. \tag{25}$$

We note that there is no dependence on the stochastic intensity variable y_t at this level of approximation.

Continuing to first order, we find we must solve

$$\mathcal{L}[h_1] = \tilde{\lambda}(t)\left(e^{y-\frac{1}{2}I_\lambda(0,t)} - 1\right)\left(h_0 - \tfrac{\partial h_0}{\partial x}\right).$$

Proceeding as before we obtain

$$h_1(x_t, y_t, t) = B(t,T)\mathcal{E}(x_t)\int_t^T \tilde{\lambda}(u)\left(\mathcal{E}(y_t\phi(t,u))\, e^{I_R(t,u)} - 1\right)\frac{\partial}{\partial x}N(d_1(x_t + I_R(t,u), t, T))\, du. \tag{26}$$

4 Conclusion

Substituting the above expressions into (17), setting $t = 0$ and $x_0 = y_0 = 0$ and resorting to unscaled variables, we obtain our main result:

Theorem 4.1 *The value of an embedded option for the holder to convert a bond to M shares of the issuer's stock per unit unit notional at maturity T is, under the assumption that the evolution of the credit default intensity and the share price are given respectively by* (1) *and* (2)*, given asymptotically in the limit of small credit default intensity by*

$$f_{option}(0) \sim B(0,T)\Bigg(MF(T)\Bigg(N(d_1(0,0,T)) + \int_0^T \frac{\overline{\lambda}(u)}{\sqrt{I_S(0,u)}}\left(e^{I_R(0,u)} - 1\right)N'(d_1(I_R(0,u),0,T)\, du\Bigg) - N(d_2(0,0,T))\Bigg) \tag{27}$$

with errors $= \mathcal{O}(\epsilon_\lambda^2)$*, where the notation is as defined in the preceding section.*

This formula is closely related to the defaultable variant of the well-known Merton (1973) formula and indeed reverts to it in the limiting case of deterministic credit intensity or when $\rho_{\lambda S} = 0$. The difference is in the inclusion of the integral term in (27). Somewhat counter-intuitively, the impact of the expected negative correlation between equity price and credit spread is seen to result in a slight *decrease* in the option value: the obvious impact of increased survival probability associated with higher-than-expected values of S_T is more than offset by the concomitant *decrease* in the jump-compensating drift (the λ_t term in (2)).

Based on the high degree of accuracy found by Turfus (2016b) in a similar calculation using a similar approximation, we expect the error associated with the result in Theorem 4.1 to be at most a few bp per unit notional.

References

Ayache, E., K. R. Vetzal and P. A. Forsyth (2002) 'Next Generation Models for Convertible Bonds with Credit Risk' Wilmott Magazine (December), pp. 68–77.

Black, F., P. Karasinski (1991) 'Bond and Option Pricing when Short Rates are Lognormal' Financial Analysts Journal, Vol. 47(4), pp. 52-59.

Davis, M. and F. R. Lischka (1999) 'Convertible Bonds with Market Risk and Credit Risk' `http://www.defaultrisk.com/pp_price_20.htm` (presented at IMS Workshop on Applied Probability, May 31–June 12, 1999, Institute of Mathematical Sciences at the Chinese University of Hong Kong).

Hull, J., A. White (1990) 'Pricing Interest Rate Derivative Securities', The Review of Financial Studies, Vol. 3, pp. 573-592.

Merton, R. C. (1973) 'Theory of Rational Option Pricing', Bell Journal of Economics and Management Science, Vol. 4(1), pp. 141–183.

Turfus, C. (2016a) 'Analytic Calibration of Black-Karasinski Short Rate Model for Low Rates' `https://archive.org/details/BlackKarasinskiModelLowRates`.

Turfus, C. (2016b) 'Contingent Convertible Bond Pricing with a Black-Karasinski Credit Model ' `https://archive.org/details/CocoBondPricingBlackKarasinski`.

Turfus, C., A. Shubert (2017) 'Analytic Pricing of CoCo Bonds' International Journal of Theoretical and Applied Finance, Vol. 20, No. 5, https://doi.org/10.1142/S0219024917500340. See also preprint at `https://archive.org/details/AnalyticPricingCoCoBonds`.